Recall Divergence Thrm

If $R$ is "nice" and $\partial R$ is also nice and $\vec{r}$ is a v.f. on $\mathbb{R}^3$ w/ cts partial derivs (i.e. components of $\vec{F}$ have cts partials), then

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$$

Ex. Calculate flux of $\vec{F} = \langle 3x, xy, 2xz \rangle$ across the surface of the cube $R = [0,1]^3$

Sol: Apply div. thrm: $\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$

$$= \iiint_R (3 + x + 2x)\, dV = \iiint_R (3+3x)\, dV$$

$$= \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} (3+3x)\, dx\, dy\, dz = \int_{z=0}^{1} \int_{y=0}^{1} \left[ 3x + \frac{3x^2}{2} \right]_0^1 dy\, dz$$

$$= \frac{9}{2} \int_{z=0}^{1} \int_{y=0}^{1} 1\, dy\, dz = \frac{9}{2} \operatorname{Area}([0,1]^2) = \frac{9}{2} \quad \blacksquare$$

Ex: Calculate flux of $\vec{F} = \langle x^2yz, xy^2z, xyz^2 \rangle$ across the boundary of the rectangular box $R = [0,a] \times [0,b] \times [0,c]$ for constants $a,b,c > 0$

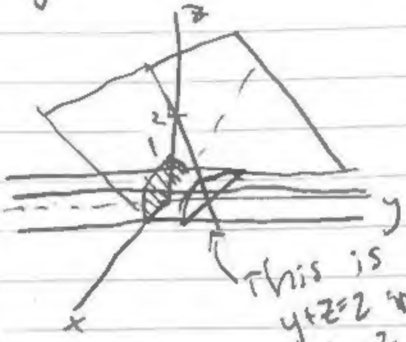
Sol: Apply div. thrm: $\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$

$$= \iiint_R (2xyz + 2xyz + 2xyz)\, dV = 6 \iiint_R xyz\, dV$$

$$= \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} xyz\, dx\, dy\, dz = \left( \int_{z=0}^{c} z\, dz \right) \left( \int_{y=0}^{b} y\, dy \right) \left( \int_{x=0}^{a} x\, dx \right) = \frac{6}{8} \left[ z^2 \right]_0^c \left[ y^2 \right]_0^b \left[ x^2 \right]_0^a$$

$$= \frac{3}{4} a^2 b^2 c^2 \quad \blacksquare$$

ex: Calculate flux ($\iint_S \vec{F}\cdot d\vec{S}$) of $\vec{F} = \langle xy, y^2+e^{xz^2}, \sin(xy)\rangle$ across the surface bounding the region w/ $z=1-x^2$, $z=0$, $y=0$, $y+z=2$

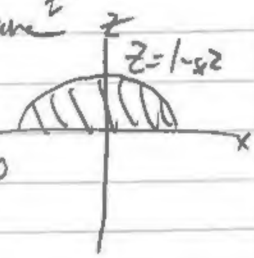



Sol: Use div thrm:

$\iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial R} \vec{F}\cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\,dS$

$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial y}[y^2+e^{xz^2}] + \frac{\partial}{\partial z}[\sin(xy)]$

$= y + 2y + 0 = \underline{3y}$  Parameterizing R: Shadow in xz plane




$\therefore R = \{(x,y,z) : -1\le x\le 1,\ 0\le z\le 1-x^2,\ 0\le y\le 2-z\}$

$\therefore \iint \vec{F}\cdot d\vec{S} = \iiint_R 3y\,dV$

$= \int_{x=-1}^{1}\int_{z=0}^{1-x^2}\int_{y=0}^{2-z} 3y\,dy\,dz\,dx = \int_{x=-1}^{1}\int_{z=0}^{1-x^2}\left[\frac{3y^2}{2}\right]_0^{2-z} dz\,dx$  $1-x^2=0$, $x=\pm 1$

$= \frac{3}{2}\int_{x=-1}^{1}\int_{z=0}^{1-x^2}(2-z)^2\,dz\,dx = \frac{3}{2}\int_{x=-1}^{1} -\frac{1}{3}\left[(2-z)^3\right]_{z=0}^{1-x^2} dx$

$= -\frac{1}{2}\int_{x=-1}^{1}\left((2-(1-x^2))^3-(2-0)^3\right)dx = -\frac{1}{2}\int_{x=-1}^{1}(1+x^2)^3-8\,dx = -\frac{1}{2}\int_{x=-1}^{1}(x^6+3x^4+3x^2-7)\,dx$

$= -\frac{1}{2}\left[\frac{x^7}{7}+\frac{3x^5}{5}+x^3-7x\right]_{-1}^{1} = -\left(1+\frac{3}{5}+\frac{1}{7}-7\right)$ ■

ex: Compute flux of $\vec{F} = \langle xye^z, xy^2z^3, -ye^z\rangle$ across S surface of box bounded by coordinate planes and $x=3$, $y=2$, $z=1$

Sol: Apply divg. thrm  note: $S=\partial R$ for $R=[0,3]\times[0,2]\times[0,1]$

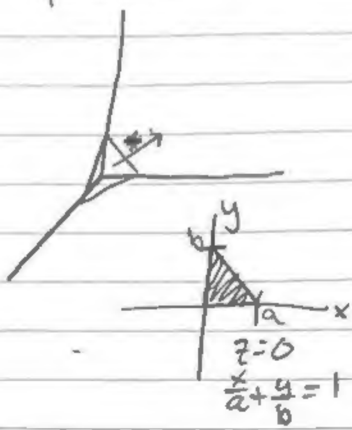
and $\operatorname{div}(\vec{F}) = ye^z + 2xyz^3 - ye^z = 2xyz^3$ ← separable

$\iint_S \vec{F}\cdot d\vec{S} = \iiint_R 2xyz^3\,dV = \left(\int_{x=0}^{3}2x\right)\left(\int_{y=0}^{2}y\,dy\right)\left(\int_{z=0}^{1}z^3\,dz\right) = 2\left[\frac{1}{2}x^2\right]_0^3\left[\frac{y^2}{2}\right]_0^2\left[\frac{z^4}{4}\right]_0^1$

$= \frac{1}{8}\left((3^2)(2^2)(1^2)\right) = \frac{9}{2}$ ■

Ex: Compute flux of $\vec{F} = \langle z, y, zx \rangle$ across the surface of tetrahedron enclosed by coordinate planes and plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ for constant $a, b, c > 0$

picture




$\vec{n} \cdot (\vec{x} - \vec{p}) = 0$ $\quad \underbrace{\langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \rangle}_{\vec{n}} \cdot \langle x, y, z \rangle = 1$

$\vec{n} \cdot \vec{x} = a$

$\therefore$ normal vector $\vec{n} = \langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \rangle$

Parameterization of tetrahedron:

$R = \{ (x, y, z) : 0 \le x \le a, 0 \le y \le b(1 - \frac{x}{a}), 0 \le z \le c(1 - \frac{x}{a} - \frac{y}{b}) \}$

And $\operatorname{div}(\vec{F}) = 0 + 1 + x = 1 + x$

$\therefore$ by divg. thm:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$$

$$= \int_{x=0}^{a} \int_{y=0}^{b(1-\frac{x}{a})} \int_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} (1+x)\, dz\, dy\, dx = \int_{x=0}^{a} \int_{y=0}^{(\ldots)} \Big[ z \Big]_0^{c(1-\frac{x}{a}-\frac{y}{b})} = \int_{x=0}^{a} (1+x) \int_{y=0}^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

$$= c \int_{x=0}^{a} (1+x) \left[ y - \frac{xy}{a} - \frac{y^2}{2b} \right]_0^{b(1-\frac{x}{a})} = c \int_{x=0}^{a} (1+x) \left[ \left(b\left(1 - \frac{x}{a}\right)\right)\left(1 - \frac{x}{a} - \frac{\left(b\left(1-\frac{x}{a}\right)\right)}{2b}\right) \right] dx$$

$y\left(1 - \frac{x}{a} - \frac{y}{2b}\right)$

$$= \frac{1}{2}bc \int_{x=0}^{a} (1+x)\left(1 - \frac{x}{a}\right)^2 dx = \frac{1}{2}bc \int_{x=0}^{a} \left(1 + \left(1 - \frac{1}{a} - \frac{1}{a}\right)x + \left(-\frac{1}{a} - \frac{1}{a} + \frac{1}{a^2}\right)x^2 + \frac{1}{a^2}x^3\right) dx$$

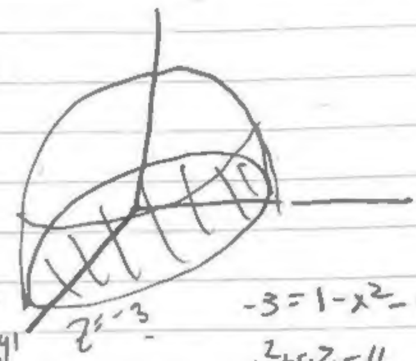
$$= \frac{1}{2}bc \int_{x=0}^{a} \left(1 + \left(1 - \frac{2}{a}\right)x + \left(\frac{1}{a^2} - \frac{2}{a}\right)x^2 + \frac{x^3}{a^2}\right) dx = \frac{1}{2}bc \left[ x + \frac{1}{2}\left(1 - \frac{2}{a}\right)x^2 + \frac{1}{3}\left(\frac{1}{a^2} - \frac{2}{a}\right)x^3 + \frac{1}{4a^2}x^4 \right]_0^a$$

$$= \frac{1}{2}bc\left(a + \frac{a^2}{2}\left(1 - \frac{2}{a}\right) + \frac{a^3}{3}\left(\frac{1}{a^2} - \frac{2}{a}\right) + \frac{1}{4}a^2\right) - 0$$

$$= \frac{1}{2}abc\left(1 + \frac{1}{2}(a-2) + \frac{1}{3}(1 - 2a) + \frac{1}{4}a\right) = \frac{1}{2}abc\left(\frac{1}{3} + a\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)\right) \quad \blacksquare$$

Ex: Compute the flux of $\vec{F} = \langle 2x^3+y^3, y^3+z^3, 3y^2z \rangle$ across the surface of the region bounded by the paraboloid $z = 1-x^2-y^2$ and the plane $z=-3$

Pic




Sol: Apply div thrm:

$\mathrm{div}(\vec{F}) = 6x^2+3y^2+3y^2 = 6x^2+6y^2 = 6(x^2+y^2)$

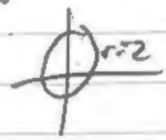
Parameterize R:

$R_{cyl} = \{(r,\theta,z) : 0 \le r \le 2, 0 \le \theta \le 2\pi, -3 \le z \le 1-r^2\}$

$-3 = 1-x^2-y^2$

$x^2+y^2=4$

Use $\begin{cases} x = r\cos(\theta) \\ y = r\sin\theta \\ z = z \end{cases}$




$\therefore \iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \mathrm{div}(\vec{F})\, dV = \iiint_{R_{cyl}} \mathrm{div}(\vec{F})_{cyl}\, r\, dV_{cyl}$

$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=-3}^{1-r^2} 6r^2 r\, dz\, dr\, d\theta$

$= (2\pi - 0) \int_{r=0}^{2} 6r^3 [z]_{-3}^{1-r^2} dr = 12\pi \int_{r=0}^{2} r^3 (1-r^2-(-3))$

$= 12\pi \int_{r=0}^{2} r r^2 (4-r^2) dr \quad \begin{matrix} u = 4-r^2 \\ du = -2r\,dr \end{matrix} \quad = -6\pi \int_{u=0}^{4} (4-u) u\, du = 6\pi \left(2u^2 - \frac{1}{3}u^3\right)\Big|_{u=0}^{4}$

$= 6\pi \left(32 - \frac{64}{3}\right)$ ☐